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A GENERALIZATION OF THE METHOD OF AVERAGING FOR THE OPTIMAL CONTROL OF NON-LINEAR OSCILLATIONS†

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The problem of the optimal control of oscillations using small control inputs is considered. It is assumed that in the first approximation of the averaging method there is no change in the slow phase vector. A second-order averaging scheme is developed, which enables the control problem to be solved over a time interval of length inversely proportional to the square of a small parameter, that is, over an "elongated time interval". Error estimates are obtained with respect to the phase trajectory, boundary conditions, functional, and control. Results are presented for a special case – a linear-quadratic control problem with periodic coefficients. The control of the phase and amplitude of non-linear oscillating systems is considered in model examples.

1. STATEMENT OF THE PROBLEM

The method of averaging [1, 2] is an effective tool for investigating weakly controllable oscillating systems [3]. However, some problems of dynamics and control often lead to a non-standard situation, in which the system of equations for the osculating variables has the form

$$\dot{x} = \varepsilon X(t, x) + \varepsilon^2 F(t, x, u), \quad x(t_0) = x^0, \quad |\varepsilon| \leq \varepsilon_0 \tag{1.1}$$

where X and F are 2π-periodic piecewise continuous functions of the argument (time time) t, $t \geq t_0$, which are sufficiently smooth with respect to the phase n_x-vector x and the control n_u-vector u; they may also depend on the small parameter ε, but, for brevity, this will not be indicated. The domains of admissible values $x \in D_x$ and $u \in D_u$ may be bounded or unbounded, open or (and) closed [1–3].

It is well known that in the first approximation of the averaging method, the evolution of the slow vector x is determined by the properties of the first term εX in system (1.1). There will be a significant change $\Delta x = |x - x^0| \sim 1$, for a change $\Delta t = t - t_0 \sim 1/\varepsilon$ in the argument, if the average $\langle X \rangle$ with respect to t of the function X does not vanish identically (and in that case $\langle X(t, x^0) \rangle \neq 0$). Otherwise, $\Delta x \sim \varepsilon$ for $\Delta t \sim 1/\varepsilon$.

Consider control system (1.1) with the condition $\langle X \rangle \equiv 0$. Simple estimates will then show that in the general case the variable x may experience a change sufficiently significant to be of interest over an "elongated" time interval $\Delta t \sim \varepsilon^{-2}$. The standard procedure of the averaging method is not applicable in such an interval; the situation demands a modified approach, which may be formulated and justified using the methods of multiscale expansions [4] or averaging [5]. Such a procedure has been presented and tested for a Cauchy problem [5] by solving meaningful problems of mechanics for the case of an uncontrollable system (1.1) in the case when $u \equiv 0 \in D_u$. It is also applicable if the control function $u = u^*(t, x)$, 2π-periodic with respect to t and smooth with respect to $x \in D_x$, is known and given.

We will give a brief outline of a scheme for constructing a solution to the first approximation (with error $O(\varepsilon)$) of degree two (over a time interval $\Delta t \sim \varepsilon^{-2}$).

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The second-order averaging scheme in [5] prescribes a standard change of the variable x to y, which is almost identical [1–6], and reduction of the Cauchy problem (1.1) to a special form with the coefficient ε^2 on the right-hand side of the equation for y:

$$
\begin{aligned}
&x = y + \varepsilon X^0(t, y), \quad X^0(t, y) \equiv \int_{t_0}^{t} X(s, y)ds, \quad X^0(t + 2\pi, y) \equiv X^0(t, y) \\
&\dot{y} = \varepsilon^2 Y(t, y) + \varepsilon^3 W(t, y), \quad y(t_0) = x^0, \quad y \in D_x \\
&Y(t, y) \equiv X'_y(t, y)X^0(t, y) + F^*(t, y), \quad F^*(t, y) \equiv F(t, y, u^*(t, y))
\end{aligned}
\tag{1.2}
$$

The functions Y and W are obtained in the standard way by differentiating the substitution (1.2) with respect to t with reference to Eq. (1.1) at $F = F^*$. The very cumbersome expression for the function W in terms of X, X^0 and F^* will not be presented here. It is 2π-periodic with respect to t, sufficiently small and bounded as a function of $y \in D_x$, and may be a continuous function of ε.

We will now consider the Cauchy problem (1.2) in the interval $\Delta t \sim \varepsilon^{-2}$. It has been shown [5] that the evolution of the system is determined by a truncated averaged equation and an initial condition of the form

$$
\xi^{\cdot} = Y_0(\xi), \quad \xi(\tau_0) = x^0, \quad \tau = \varepsilon^2 t, \quad \Delta\tau = \tau - \tau^0 \sim 1, \quad Y_0(y) \equiv \langle Y(t, y) \rangle \tag{1.3}
$$

The dot denotes differentiation with respect to the slow argument ("time") τ. System (1.3) lends itself much more easily than (1.2) to an analytical and numerical treatment. It does not contain the time explicitly (is autonomous), and its solution $\xi \in D_x$ depends on $\Delta\tau$. Of course, it is assumed here that the function $Y_0(\xi)$ is fairly easy to construct.

It has been proved [5] that the solutions of the Cauchy problems (1.1)–(1.3) are ε-close together over the relevant elongated time interval $\Delta t \sim \varepsilon^{-2}$:

$$
\begin{aligned}
&\left|x(t, t_0, x^0, \varepsilon) - \xi(\Delta\tau, x^0)\right| \le \varepsilon C; \quad 0 \le \Delta t \le L\varepsilon^{-2}, \quad x, y, \xi \in D_x \\
&\left|y(t, t_0, x^0, \varepsilon) - \xi(\Delta\tau, x^0)\right| \le \varepsilon C; \quad L, C = \text{const}
\end{aligned}
\tag{1.4}
$$

The constant C may be determined effectively [5] using Gronwall's Lemma.

The approximate solution ξ is improved in the standard way, using the procedure of the averaging method and the explicit expression for the function W. This approach, however, requires that the functions X and F^* should be smooth to a higher order. Henceforth, we shall limit ourselves to constructing and investigating a solution ξ of the first approximation of degree two, that is, with an error $O(\varepsilon)$ over the elongated interval $\Delta t \sim \varepsilon^{-2}$.

The procedure to be described may be used in a natural way to find approximate solutions of optimal control problems for the motions of the oscillating system (1.1) which, within the context of the standard approach [3], is weakly controllable: $\Delta x \sim \varepsilon$ if $\Delta t \sim \varepsilon^{-1}$ for bounded controls u, $|u| \sim 1$. We will consider formulations, of interest in an applied setting, of problems with an integral functional J over a fixed time interval $t \in [t_0, t_f]$:

$$
\begin{aligned}
&J[u] \to \min_u, \quad u \in U \\
&J = J_1[u] \equiv g(x(t_f)) + \varepsilon^2 \int_{t_0}^{t_f} G(t, x, u)dt \quad \text{(Problem 1)} \\
&g(x(t_f)) = 0, \quad J = J_2[u] \equiv \varepsilon^2 \int_{t_0}^{t_f} G(t, x, u)dt, \quad t_f = \frac{\Theta}{\varepsilon^2} \quad \text{(Problem 2)} \\
&\varepsilon^2 t_0 = \tau_0 \sim 1
\end{aligned}
\tag{1.5}
$$

where, in both problems, G is a scalar function, 2π-periodic in t and sufficiently smooth in x and u. The coefficient ε^2 of the integrals in (1.5) is a normalization factor: if $G \sim 1$, the corresponding terms will also be quantities of the order of unity. The function g – a scalar in Problem 1 and an n_2-vector in Problem 2

– characterizes the conditions imposed on the final values $x(t_f)$ of the phase vector and is assumed to be sufficiently smooth. In particular, g defines the measure of the required value $x(t_f) = x^f$ (where x^f is given)

$$
\begin{aligned}
& g \equiv (x(t_f) - x^f)^T N_1 (x(t_f) - x^f) \text{ in Problem 1} \\
& g \equiv N_2(x(t_f) - x^f) \text{ in Problem 2}
\end{aligned}
\tag{1.6}
$$

where N_1 is an $n_x \times n_x$ (symmetric) matrix, assumed to be positive semi-definite and N_2 is an $n_2 \times n_x$ matrix, where $1 \le n_2 \le n_x$; if $n_2 = n_x$ and $\det N_2 \ne 0$, then by (1.6) the second condition in (1.5) takes the form $x(t_f) = x_f$, corresponding to a two-point control problem [3, 6].

Thus, we are going to consider optimal control problems (1.1), (1.5). Our aim will be to construct approximate solutions for which the errors in the trajectory x and the functional J are at most $O(\varepsilon)$. A smooth (not bang-bang) control u must be constructed with error $O(\varepsilon)$ in an open-loop, $u_p(t)$, or feedback, $u_x(t, x)$, form. In the case of a bang-bang control, an error should be defined in terms of an integral metric or a suitable functional [3].

2. APPROXIMATE SOLUTION OF OPTIMAL CONTROL PROBLEMS

We shall use the necessary conditions for optimality in the form of the maximum principle [6] (the procedure will be analogous to what was done in [3]):

$$
\begin{aligned}
H &= \varepsilon(X, p) + \varepsilon^2[(F, p) - G] \to \max_u, \quad u \in U \\
u &= u^*(t, x, p), \quad H^* = \varepsilon(X, p) + \varepsilon^2[(F^*, p) - G^*]
\end{aligned}
\tag{2.1}
$$

where H is the Hamiltonian of the control system (1.1), (1.5) and p is a variable (momentum) conjugate to the phase vector x. The function u^*, defined by the maximum condition (2.1), is a 2π-periodic or periodically continuable function of t. Smoothness with respect to x and p is assumed in a certain domain $(x, p) \in D_x \times D_p$; with respect to t, it is sufficient to assume piecewise smoothness or continuity [3].

The initial phase variable x and the momentum p are found by solving the boundary-value problem of the maximum principle with conditions corresponding to (1.1), (1.5) at $t = t_0, t_f$:

$$
\begin{aligned}
\dot{x} &= \varepsilon X(t, x) + \varepsilon^2 F^*(t, x, p), \quad x(t_0) = x^0 \\
\dot{p} &= -\varepsilon(X, p)'_x - \varepsilon^2[(F, p)'_x - G'_x]^*, \quad t_0 \le t \le t_f = \Theta\varepsilon^{-2}
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
& p(t_f) = -g'_x(x(t_f)) \text{ in Problem 1} \\
& g(x(t_f)) = 0, \quad p(t_f) = (\lambda, g(x(t_f)))'_x \text{ in Problem 2}
\end{aligned}
\tag{2.3}
$$

where λ is an n_2-vector of Lagrange multipliers, which are computed together with the $2n_x$ integration constants of the Hamiltonian system (2.2).

The investigation of boundary-value problems (2.2), (2.3) is obviously extremely difficult from both the analytical and numerical standpoints.

Since the right-hand sides of Eqs (2.2) are periodic in t, one can apply a change-of-variables procedure analogous to (1.2), (1.3) (the averaging method). Since by assumption $\langle X \rangle \equiv 0$, it follows that, subject to the smoothness conditions, $\langle (X, p)'_x \rangle \equiv 0$. By a simultaneous change of variables $(x, p) \to (y, q)$ of the form (1.2)

$$
x = y + \varepsilon X^0(t, y), \quad p = q - \varepsilon(q, X_y^{0\prime}(t, y))
\tag{2.4}
$$

one can reduce Eqs (2.2) to a form analogous to (1.3):

$$
\begin{aligned}
\dot{y} &= \varepsilon^2 Y(t, y, q) + \varepsilon^3 \Delta Y, \quad Y \equiv X'_y X^0 + F^* \\
\dot{q} &= \varepsilon^2 Q(t, y, q) + \varepsilon^3 \Delta Q, \quad Q \equiv -(q, X''_{y^2} X^0) + (q, X^0)'_y - [(q, F)'_y - G'_y]^*
\end{aligned}
\tag{2.5}
$$

Henceforth the terms $O(\varepsilon^3)$ in (2.5) will be omitted, since they lead to $O(\varepsilon)$ errors in the solution for $\Delta t \sim \varepsilon^{-2}$. This assumption is certainly valid if the functions Y and Q satisfy Lipschitz conditions, and ΔY and ΔQ are uniformly bounded in some domain $y \in D_x$, $q \in D_p$. The error is estimated using an integral inequality (Gronwall's Lemma). Note that the change of variables (2.4) and system (2.5) are not canonical.

The approximate procedure of the averaging method consists in dropping $O(\varepsilon^3)$ perturbations and averaging the functions Y and Q over the explicitly occurring time t. This operation is analogous to the change of variables [5].

$$\begin{aligned} y &= \xi + \varepsilon^2 \int_{t_0}^{t} [Y(t', \xi, \eta) - Y_0(\xi, \eta)]dt', \quad Y_0 = \langle Y \rangle \\ q &= \eta + \varepsilon^2 \int_{t_0}^{t} [Q(t', \xi, \eta) - Q_0(\xi, \eta)]dt', \quad Q_0 = \langle Q \rangle \end{aligned} \tag{2.6}$$

in Eqs (2.5) and dropping $O(\varepsilon^3)$ terms in the system of equations for ξ and η. The result is an averaged system of equations that can be used to eliminate the coefficient ε^2:

$$\xi^{\cdot} = Y_0(\xi, \eta), \quad \eta^{\cdot} = Q_0(\xi, \eta), \quad \tau = \varepsilon^2 t, \quad \tau_0 \le \tau \le \Theta \tag{2.7}$$

The autonomous system (2.7) must be integrated over an interval in which the change in the slow time is of the order of unity, which significantly simplifies the computations. Note that, although the truncated system (2.5) is not Hamiltonian, the averaged system (2.7) is canonical, with Hamiltonian

$$H_0(\xi, \eta, \varepsilon) = \varepsilon^2 h(\xi, \eta), \quad h \equiv (\eta, \langle X'_\xi X^0 + F^* \rangle) - \langle G^* \rangle \tag{2.8}$$

This property is established by constructing the canonical averaging scheme of the second approximation with respect to ε for system (2.2) [3]. It may also be obtained by a preliminary transformation, similar to (1.2), of the variable x in the initial optimal control problem (1.1), (1.5) (see below, formulae (2.12)–(2.14)).

Thus, we can proceed to integrate – analytically or numerically – the system of canonical equations (2.7), which admits of an integral h = const (see (2.8)). The solution $\xi(\tau, x^0, p^0)$, $\eta(\tau, x^0, p^0)$ corresponding to a family of Cauchy problems ($p^0 \in D_p$ is an unknown parameter), provided that $(\xi, \eta) \in D_x \times D_p$ for $\tau_0 \le \tau \le \Theta$, satisfies ε-closeness conditions like (1.4)

$$\begin{gathered} (|x - \xi| + |p - \eta|) \le C\varepsilon; \quad t_0 \le t \le \Theta\varepsilon^{-2} \\ (|y - \xi| + |q - \eta|) \le C\varepsilon; \quad \Theta, C = \text{const} \end{gathered}$$

With x^0 fixed, if a family of solutions ξ, η depending on the parameter p^0 has been constructed, then the unknowns p^0 or (p^0, λ) defining the solutions of boundary-value Problems 1 and 2, respectively, are found from the system of finite equations

$$\begin{aligned} &\eta(\Theta, x^0, p^0) = -q'_x(\xi(\Theta, x^0, p^0)), \quad p^0 = p^0(\Theta, x^0) \text{ in Problem 1} \\ &g(\xi(\Theta, x^0, p^0)) = 0, \quad \eta(\Theta, x^0, p^0) = (\lambda, g(\xi(\Theta, x^0, p^0))'_x \text{ in Problem 2} \\ &p^0 = p^0(\Theta, x^0), \quad \lambda = \lambda(\Theta, x^0) \end{aligned} \tag{2.9}$$

It is assumed that the roots p^0 or (p^0, λ) of Eqs (2.9) have been determined and are simple, that is, the Jacobians of the implicit functions for the x^0 and Θ under consideration do not vanish for these values. Proceeding then as in [3], one can state the following

Theorem. The solutions of boundary-value problems (2.2), (2.3) and (2.7), (2.9) are ε-close together with respect to trajectories and boundary conditions. The stationary values of the functional J (1.5) will also be close together if x and u are replaced by the following approximations

$$x = \xi(\tau, x^0, p^0), \quad u = u^*(t, \xi, \eta) \tag{2.10}$$

We have

$$|J[u^*(t, x, p)] - J[u^*(t, \xi, \eta)]| \le C\varepsilon$$

If there are several roots p^0 or (p^0, λ), the selection criterion is the value of the functional J. The solutions of the exact and approximate problems will be close together with respect to controls if u^* is uniformly smooth as a function of x and p.

Examples of such formulations are linear-quadratic problems whose coefficients are 2π-periodic functions of t, with no constraints on the control [3] (see Section 3 and the examples in Section 4). The expressions for u^* in (2.10) are 2π-periodic functions of t and, in a certain way, of the slow argument $\tau = \varepsilon^2 t$, through the functions ξ and η. Fully expressed in terms of the arguments and parameters introduced above, these functions are an open-loop control u_p and a feedback control u_s

$$\begin{aligned} u^* &= u_p(t, \tau-\tau_0, \Theta-\tau_0, x^0), \quad t_0 \le t \le \Theta\varepsilon^{-2} \\ u^* &= u_s(t, 0, \Theta-\tau, x), \quad t_0 \le t < \Theta\varepsilon^{-2} \end{aligned} \tag{2.11}$$

Controls (2.11) have pronounced resonant properties with respect to the initial (fast) time t [3].

The Hamiltonian system (2.2) may be dealt with by a canonical change of variables $(x, p) \to (y, q)$ [2, 3] instead of the standard type (2.4). As the construction of the generating function and the substitution formula is extremely laborious, the approach described in Section 1, involving the change of a variables (1.2), seems preferable. In the control problem, Eq. (1.1) is reduced to the form

$$\dot{y} = \varepsilon^2 X_y'(t, y)X^0(t, y) + \varepsilon^2 F(t, y, u) + \varepsilon^3 W(t, y, u) \tag{2.12}$$

with corresponding transformations in the boundary conditions and functionals (1.5). In the first approximation with respect to ε, they retain their old form, in which the variable x is replaced by y. Evaluating the scalar product of the right-hand side of Eq. (2.12) by the adjoint vector q, subtracting the function $G(t, y, u)$ and maximizing with respect to $u \in U$, one obtains in the first approximation an expression for u^* which is identical with (2.1) but with $x \to y, p \to q$. Thus, the system with the function

$$\begin{aligned} H^*(t, y, q, \varepsilon) &= \varepsilon^2(q, (X_y'(t, y)X^0(t, y) + F^*(t, y, q))) - \varepsilon^2 G^*(t, y, q) \\ F^* &= F(t, y, u^*), \quad G^* = G(t, y, u^*), \quad u^* = u^*(t, y, q) \end{aligned} \tag{2.13}$$

will be Hamiltonian. In the first approximation, the averaged Hamiltonian has the form (2.8):

$$\langle H^* \rangle = \varepsilon^2 h, \quad h = (q, (\langle X_y' X^0 \rangle + \langle F^* \rangle)) - \langle G^* \rangle, \quad h = \text{const} \tag{2.14}$$

It is constant along trajectories of the averaged system. In control problems for mechanical oscillating systems, explicit dependence of the function u^* (2.1) on the time (or phase) t is achieved, as a rule, through the function F (1.1). The integrand G in (1.5) may not depend explicitly on t; this is usually the case in applications (see the examples in Section 4).

The approach outlined above considerably extends the range of problems of dynamics and control for oscillating systems [3, 5] that can be handled by asymptotic methods. Together with the second-order averaging scheme, one can also consider higher-order schemes [5], in which the perturbations and controls are of significantly different orders of magnitude with respect to ε. The problems are still of considerable interest from the standpoint of the application of asymptotic methods in the case when the functions F and G do not depend explicitly on the time t. By analogy with the approach outlined above, one can investigate problems in which the completion time t_f of the control process is not fixed. In such cases the formulations of the problem must correspond to the interval $\Delta t \sim \varepsilon^{-2}$ under consideration. For example, in time-optimal control problems it is assumed that $F \sim \varepsilon^2 u$, where the control u is confined to a bounded domain U.

3. LINEARLY QUADRATIC OSCILLATION-CONTROL PROBLEMS

Let us consider a linear analogue of control system (1.1), with appropriate boundary conditions and integral quadratic functionals $J_{1,2}$ (1.5), (1.6) of the form

$$\dot{x} = \varepsilon A(t)x + \varepsilon^2[C(t)x + B(t)u], \quad x(t_0) = x^0, \quad |u| < \infty$$

$$J_1[u] = \frac{1}{2}(x - x^f)^T N_1(x - x^f)\big|_{t_f} + \frac{\varepsilon^2}{2}\int_{t_0}^{t_f} u^T G(t)u dt \quad \text{in Problem 1}$$

$$N_2(x - x^f)\big|_{t_f} = 0, \quad J_2[u] = \frac{\varepsilon^2}{2}\int_{t_0}^{t_f} u^T G(t)u dt \quad \text{in Problem 2} \tag{3.1}$$

$$t_0 \le t \le t_f = \Theta\varepsilon^{-2}$$

where G is a positive-definite (symmetric) $n_u \times n_u$ matrix and B is an $n_x \times n_u$ matrix. The matrices A, B, C and G are 2π-periodic functions of time. As in Sections 1 and 2, it is assumed that $\langle A\rangle = 0$, that is, $A^0(t)$ is a 2π-periodic matrix.

We will first use the approach of Section 1, which is exact with respect to powers of ε and requires a preliminary vector transformation $x \to y$. We again obtain linearly quadratic control problems

$$x = [E + \varepsilon A^0(t)]y, \quad A^0 = \int_{t_0}^{t} A(t')dt', \quad y(t_0) = x(t_0) = x^0$$

$$\dot{y} = \varepsilon^2 C_\varepsilon(t)y + \varepsilon^2 B_\varepsilon(t)u, \quad y^f = [E + \varepsilon A^0(t_f)]^{-1}x^f$$

$$C_\varepsilon(t) = (E + \varepsilon A^0)^{-1}(AA^0 + C), \quad B_\varepsilon(t) = (E + \varepsilon A^0)^{-1}B \tag{3.2}$$

$$J_1[u] = \frac{1}{2}(y - y^f)^T N_1^\varepsilon(y - y^f)\big|_{t_f} + \frac{\varepsilon^2}{2}\int_{t_0}^{t_f} u^T G(t)u dt \quad \text{in Problem 1}$$

$$N_2^\varepsilon(y - y^f)\big|_{t_f} = 0, \quad J_2[u] = \frac{\varepsilon^2}{2}\int_{t_0}^{t_f} u^T G(t)u dt \quad \text{in Problem 2}$$

The matrices $N_{1,2}^\varepsilon$ are obtained by standard algebraic operations and depend on the parameters ε and t_f; moreover $N_{1,2}^\varepsilon = N_{1,2} + O(\varepsilon)$.

Problems (3.2) will be dealt with by applying the necessary optimality conditions in the form of the maximum principle [3, 6]. By analogy with (2.1), we obtain exact expressions

$$u = u^*(t, y, q, \varepsilon) \equiv G^{-1}(t)B_\varepsilon^T(t)q, \quad H_\varepsilon^*(t, y, q, \varepsilon) = \varepsilon^2 h_\varepsilon(t, y, q)$$

$$h_\varepsilon \equiv q^T C_\varepsilon(t)y + \frac{1}{2}q^T R_\varepsilon(t)q, \quad R_\varepsilon(t) = B_\varepsilon(t)G^{-1}(t)B_\varepsilon^T(t) \tag{3.3}$$

The linear boundary-value problems of the maximum principle corresponding to problem (3.2), in formulations that are exact in powers of ε, take the form

$$\dot{y} = \varepsilon^2 C_\varepsilon(t)y + \varepsilon^2 R_\varepsilon(t)q, \quad \dot{q} = -\varepsilon^2 C_\varepsilon^T(t)q$$

$$q(t_f) = -N_1^\varepsilon(y(t_f) - y^f) \quad \text{in Problem 1} \tag{3.4}$$

$$N_2^\varepsilon(y(t_f) - y^f) = 0, \quad q(t_f) = \lambda^T N_2^\varepsilon \quad \text{in Problem 2}$$

Equation (3.4) for the momentum q is integrated independently of the phase vector y. The approximate solution ξ, η of the Hamiltonian system (3.4) over a time interval $t \sim \varepsilon^{-2}$ may be constructed to within a given accuracy in powers of ε. To within $O(\varepsilon)$, it is described by the relations

$$\xi^{\cdot} = (\langle AA^0\rangle + \langle C\rangle)\xi + \langle BG^{-1}B^T\rangle\eta$$
$$\xi(\tau_0) = x^0; \quad \eta^{\cdot} = -(\langle AA^0\rangle + \langle C\rangle)\eta \tag{3.5}$$
$$\eta(\Theta) = -N_1(\xi(\Theta) - x^f) \quad \text{in Problem 1}$$
$$N_2(\xi(\Theta) - x^f) = 0, \quad \eta(\Theta) = \lambda^T N_2 \quad \text{in Problem 2}$$

The dot denotes differentiation with respect to the slow time $\tau = \varepsilon^2 t$; the variable (argument) τ varies in the range $\varepsilon^2 t_0 = \tau_0 \le \tau \le \Theta = \varepsilon^2 t_f$.

System (3.5), which has constant coefficients, is integrated by either algebraic or numerical methods. It admits of a first integral

$$\langle h_0\rangle = \eta^T(\langle AA^0\rangle + \langle C\rangle)\xi + \frac{1}{2}\eta^T\langle B^TG^{-1}B\rangle\eta = \text{const} \tag{3.6}$$

Note that if the matrices A and A^0 commute (in particular, if they are diagonal), then $\langle AA^0\rangle = 0$. Indeed, by (3.2), we have

$$\frac{1}{2}\frac{d}{dt}(A^0A^0) = \frac{1}{2}AA^0 + \frac{1}{2}A^0A = AA^0(=A^0A)$$

which implies the assertion. The solution of problem (3.5) becomes elementary when $\langle C\rangle = 0$, since

$$\xi = x^0 + \langle R_0\rangle\eta(\tau - \tau_0), \quad \eta = \text{const} \tag{3.7}$$

For the boundary conditions to be solvable, the matrices $M_{1,2}$ defined by

$$M_1 = E + N_1\langle R_0\rangle(\Theta - \tau_0) \quad \text{in Problem 1}$$
$$M_2 = \begin{Vmatrix} N_2^T\langle R_0\rangle(\Theta - \tau_0) & 0 \\ 0 & -N_2 \end{Vmatrix} \quad \text{in Problem 2} \tag{3.8}$$

must be non-singular. By virtue of our assumptions, the matrix M_1 is symmetric and positive-definite, hence non-singular. The matrix M_2 will also be invertible if $\det(N_2^T\langle R_0\rangle N_2) \ne 0$, which is indeed assumed.

A brief outline now follows of an averaging method of the second degree applied to the initial problems (3.1). Using relations (2.1), we obtain exact expressions for the optimal control u^* and Hamiltonian H^*

$$u = u^*(t, x, p) = G^{-1}(t)B^T(t)p, \quad R(t) \equiv B(t)G^{-1}(t)B^T(t)$$
$$H^*(t, x, p, \varepsilon) = \varepsilon p^T A(t)x + \varepsilon^2\left[p^T C(t)x + \frac{1}{2}p^T R(t)p\right] \tag{3.9}$$

The boundary-value problems of the maximum principle are similar in form to (2.2)

$$\dot{x} = \varepsilon Ax + \varepsilon^2(Cx + Rp), \quad x(t_0) = x^0, \quad \dot{p} = -\varepsilon A^T p - \varepsilon^2 C^T p$$
$$p(t_f) = -N_1(x(t_f) - x^f) \quad \text{in Problem 1} \tag{3.10}$$
$$N_2(x(t_f) - x^f) = 0, \quad p(t_f) = \lambda^T N_2 \quad \text{in Problem 2}$$

Applying a transformation of type (2.4) to Hamiltonian system (3.10), one can eliminate the $O(\varepsilon)$ terms; this gives the expressions

$$x = y + \varepsilon A^0(t)y, \quad p = q - \varepsilon A^{0T}(t)q$$
$$\dot{y} = \varepsilon^2 C_\varepsilon^+ y + \varepsilon^2(E + \varepsilon A^0)^{-1}R(E - \varepsilon A^{0T})q \tag{3.11}$$
$$\dot{q} = -\varepsilon^2 C_\varepsilon^{-T} q, \quad C_\varepsilon^\pm = C_0^\pm + \varepsilon\Delta C_\varepsilon^\pm, \quad C_0^+ \ne C_0^-$$

The matrices $C_\varepsilon^\pm(t)$ are obtained by standard operations in terms of $A(t)$, $A^0(t)$, $C(t)$. As remarked in Section 1, the change of variables $(x, p) \to (y, q)$ is not canonical, either in the exact sense or in case when terms $O(\varepsilon^3)$ in the equations are ignored. Thus, considering the first-approximation systems (excluding terms $O(\varepsilon^3)$) corresponding to (3.4), (3.11), one obtains different representations. Equations (3.4) are Hamiltonian in any order of approximation with respect to powers of ε. The transformation and system (3.11), in contrast, are not canonical, because

$$C_0^+ = AA^0 + C = C_0, \quad C_0^- = -A^0A + C$$

Averaging over t, however, one can prove that $\langle C_0^+\rangle = \langle C_0^-\rangle$, which is established by integrating the expressions AA^0 or A^0A by parts. An analogous Hamiltonian property of the averaged systems in the first approximation with respect to ε was noted in Section 2 for the general case of non-linear functions X, F and G.

4. EXAMPLES

We will now consider some model optimal control problems over an elongated time interval for oscillating systems with one degree of freedom.

1. *Control of the phase of the oscillation by parametric section.* Based on the technique outlined in section 2. Let us investigate a system of the form

$$\ddot{x} + \omega^2 x = \varepsilon w(x, \dot{x}) + \varepsilon^2 f(x, \dot{x})u, \quad x(0) = x^0, \quad \dot{x}(0) = \dot{x}^0 \tag{4.1}$$

where εw is a perturbation, assumed henceforth to be of low frequency and $\varepsilon^2 fu$ is a control term, where the control u is to be determined. We may assume without loss of generality that the frequency ω is 1, and reduce Eq. (4.1) to the form of a system with rotating phase

$$\begin{aligned} &x = a\sin\psi, \quad \dot{x} = a\cos\psi, \quad a(0) = a^0, \quad \psi(0) = \psi^0 \\ &\dot{a} = \varepsilon A(a, \psi) + \varepsilon^2 K(a, \psi)u, \quad A = w\cos\psi, \quad K = f\cos\psi \\ &\dot{\psi} = 1 + \varepsilon\Psi(a, \psi) + \varepsilon^2\Lambda(a, \psi)u, \quad \Psi = -(w/a)\sin\psi, \quad \Lambda = -(f/a)\sin\psi \end{aligned} \tag{4.2}$$

For simplicity, we set $w = \gamma a$ and $f = \beta x$, where γ and β are constants. As a result, the equation for ψ in (4.2) is separated out and, after introduction of the slow variable $\varphi = \psi - t$, may be reduced to the standard form

$$\dot{\varphi} = -\varepsilon\gamma\sin(t + \varphi) - \varepsilon^2\beta u\sin^2(t + \varphi), \quad \varphi(0) = \psi^0 \tag{4.3}$$

We formulate a control problem of type (1.5) for system (4.3)

$$\begin{aligned} &J_1[u] = \frac{1}{2}k\varphi^2(t_f) + \varepsilon^2 I[u], \quad I[u] = \frac{1}{2}\int_0^{t_f} u^2 dt \quad \text{(in Problem 1)} \\ &\varphi(t_f) = 0, \quad J_2[u] = \varepsilon^2 I[u], \quad t_f = \Theta\varepsilon^{-2}, \quad |u| < \infty \quad \text{(in Problem 2)} \end{aligned} \tag{4.4}$$

where $k > 0$ is a weighting factor in the functional J_1 of (4.4) and no further (e.g. geometrical) constraints are imposed on the control u.

Applying the procedure described in Section 2 to the optimal control problem (4.3), (4.4), we obtain the control and averaged boundary-value problem in the first approximation as

$$u^* = -\beta\eta\sin^2(t + \xi), \quad \xi^\cdot = -\frac{1}{2}\gamma^2 + \frac{3}{8}\beta^2\eta, \quad \xi(0) = \psi^0 \tag{4.5}$$

where $\eta = \text{const}$. The boundary conditions imply

$$\begin{aligned} &\eta = k\left(\frac{1}{2}\gamma^2\Theta - \psi^0\right)\left(1 + \frac{3}{8}k\beta^2\Theta\right)^{-1} \quad \text{in Problem 1} \\ &\eta_{k\to\infty} \to \eta^* = \frac{1}{\beta^2\Theta}\left(\frac{4}{3}\gamma^2\Theta - \frac{8}{3}\psi^0\right), \quad \eta = \eta^*, \quad I[u^*] = \frac{1}{4}\beta^2\eta^2\Theta \quad \text{in Problem 2} \end{aligned} \tag{4.6}$$

It follows from (4.5) that the amplitude of the control u^* and the cost I of the control decrease as β increases. The effect of the low-frequency noise w is independent of the sign of γ; the noise may promote or obstruct the required variation of the phase correction φ.

Besides the above formation, one might consider another, in which the quantities φ, φ^0, $\varphi^f = \varphi(t_f)$ are defined modulo 2π, that is, φ and $\varphi \pm 2\pi$ are identified. Standard methods [3] may be used to investigate the problems with additional constraints of the type $|u| \le u_0$.

2. *Control of quasilinear oscillations.* Consider the following non-linear system with one degree of freedom [6]

$$\ddot{z} + \Phi(z) = -\Lambda\dot{z} + V, \quad z(0) = z^0, \quad \dot{z}(0) = \dot{z}^0 \tag{4.7}$$

where Φ is a non-linear increasing force, Λ is the coefficient of dissipation and V is the control. Let $\Phi(0) = 0$, $\Phi'(0) = \omega^0 > 0$; the higher-order derivatives Φ'', Φ''', ... may take arbitrary values at $z = 0$. We shall characterize the deviations z by a small parameter ε: $z = \varepsilon s$. Assuming that Λ and V are sufficiently small, we reduce system (4.7) in dimensionless time $t^* = \omega t$ (the asterisk will henceforth be omitted) to the form

$$\begin{aligned} &\ddot{s} + s = \varepsilon\alpha s^2 + \varepsilon^2\beta s^3 - \varepsilon^2\chi\dot{s} + \varepsilon^2 u + \varepsilon^3 \ldots; \quad s(0) = s^0, \quad \dot{s}(0) = \dot{s}^0 \\ &\alpha = -\frac{1}{2\omega^2}\Phi''(0), \quad \beta = -\frac{1}{6\omega^2}\Phi'''(0), \quad \varepsilon\chi = \frac{\Lambda}{\omega}, \quad \varepsilon^3 u = \frac{V}{\omega} \end{aligned} \tag{4.8}$$

The variables $s, \dot{s}$, the control u and the parameters $s^0, \dot{s}^0$, α, β, χ are assumed to be quantities of the same order of magnitude as unity. With the phase variables $s, \dot{s}$ replaced by Van der Pol osculating variables x_1, x_2, the equation of the controlled motion (4.8) may be reduced to the form of a standard system (1.1) in which

$$\begin{aligned} &s = x_1\cos t + x_2\sin t, \quad \dot{s} = -x_1\sin t + x_2\cos t \\ &X_1 = -\alpha s^2\sin t, \quad F_1 = -(\beta s^3 - \chi\dot{s} + u)\sin t, \quad x_1(0) = s^0 \\ &X_2 = \alpha s^2\cos t, \quad F_2 = -(\beta s^3 - \chi\dot{s} + u)\cos t, \quad x_2(0) = \dot{s}^0 \end{aligned} \tag{4.9}$$

Substituting the expressions from (4.9) for the variables s and $\dot{s}$ into the functions X_i and $F_i (i = 1, 2)$, it can be verified directly that $\langle X_i \rangle \equiv 0$ $(i = 1, 2)$, that is, the control system satisfies the conditions of Sections 1 and 2. We can thus formulate and investigate specific optimal control problems of type (1.1), (1.5) for the system over an interval $0 \le t \le \Theta\varepsilon^{-2}$. Such formulations may be either the build-up or extinction of oscillations $|x(t_f)| \gtrless |x(0)|$, or a simultaneous variation of amplitude and phase, that is, the components $x_1(t_f)$ and $x_2(t_f)$ must take given values.

As a simpler problem, let us consider control of the oscillation amplitude. To that end, it is convenient to replace the variables x_1, x_2 by "amplitude-phase" variables, using Bogolyubov's substitution [1–3]

$$\begin{aligned} &s = a\cos\psi, \quad \dot{s} = -a\sin\psi, \quad a > 0 \\ &\dot{a} = \varepsilon X_a(a, \psi) + \varepsilon^2 F_a(a, \psi, u), \quad a(0) = a^0 = (s^{02} + \dot{s}^{02})^{1/2} \\ &\dot{\psi} = 1 + \varepsilon X_\psi(a, \psi) + \varepsilon^2 F_\psi(a, \psi, u), \quad \psi(0) = \psi^0 \in [0, 2\pi) \\ &X_a = -\alpha s^2\sin\psi, \quad F_a = -(\beta s^3 - \chi\dot{s} + u)\sin\psi \\ &X_\psi = -\alpha s^2 a^{-1}\cos\psi, \quad F_\psi = -(\beta s^3 - \chi\dot{s} + u)a^{-1}\cos\psi \end{aligned} \tag{4.10}$$

Substitute the expressions from (4.10) for the variables s and $\dot{s}$ into the functions $X_{a,\psi}$, $F_{a,\psi}$. If $a = 0$, the substitution is singular (the phase ψ is undefined); Hence, if $a \simeq 0$, we have to apply accurate reasoning and then evaluate the relevant limits [3]. Note that for the averages with respect to ψ, we have $\langle X_{a,\psi} \rangle \equiv 0$ for all $a \ge 0$.

We can formulate a problem of type (1.5) for system (4.10), in which the amplitude a is assumed to vary in the required way

$$
\begin{aligned}
&J_1[u] = \frac{k}{2}[a(t) - a^f]^2 + \frac{\varepsilon^2}{2}\int_0^{t_f} u^2 dt, \quad u \in U \quad \text{in Problem 1} \\
&a(t_f) = a^f \geq 0, \quad J_2[u] = \frac{\varepsilon^2}{2}\int_0^{t_f} u^2 dt, \quad t_f = \frac{\Theta}{\varepsilon^2} \quad \text{in Problem 2}
\end{aligned}
\tag{4.11}
$$

Approximation solutions of the optimal control problems (4.10), (4.8) with error $O(\varepsilon)$ may be constructed quite simply, if the time argument t is replaced by the phase ψ. Dividing $\dot{a}$ by $\dot{\psi}$, we obtain an equation

$$
a' = \varepsilon X_a(a, \psi) + \varepsilon^2[-X_a(a, \psi)X_\psi(a, \psi) + F_a(a, \psi, u)], \quad a(\psi^0) = a^0 \tag{4.12}
$$

with argument ψ, $\psi^0 \leq \psi \leq \psi_f$. Here the final value of ψ_f is unknown, but it may be shown to within a relative error $O(\varepsilon^2)$ that $\psi_f = \Theta\varepsilon^{-2}$, which is admissible to within the required accuracy. Thus, t may be replaced by ψ in expressions (4.11) and the scalar problem (4.12), (4.11) for the amplitude a can be investigated using a second-degree averaging scheme (see Sections 1 and 2).

With no constraints on the control u of (2.1), we obtain $u^* = -p\sin\psi$, where p is the variable conjugate to a (the momentum). In the first approximation with respect to ε, one obtains two-point boundary-value problems:

$$
\begin{aligned}
&\xi' = -\frac{\chi\xi}{2} + \frac{\eta}{2}, \quad \eta' = \frac{\chi\eta}{2}, \quad \theta = \varepsilon^2\psi; \quad \xi(\theta_0) = a^0 \\
&\eta(\Theta) = -k(\xi(\Theta) - a^f) \quad \text{in Problem 1} \\
&\xi(\Theta) = a^f \quad \text{in Problem 2}
\end{aligned}
\tag{4.13}
$$

The prime denotes differentiation of the averaged amplitude ξ and momentum η with respect to the slow phase θ.

Solution of boundary-value problems (4.13) yields simple expressions for $\eta = \eta_{1,2}$

$$
\begin{aligned}
&\eta_1 = k\left(a^f - a^0\exp\left(-\frac{\chi}{2}(\Theta - \theta_0)\right)\right)\left(\exp\frac{\chi}{2}(\Theta - \theta_0) + \frac{k}{\lambda}\operatorname{sh}\frac{\chi}{2}(\Theta - \theta_0)\right)^{-1}\exp\frac{\chi}{2}(\theta - \theta_0) \\
&\eta_2 = \left(a^f - a^0\exp\left(-\frac{\chi}{2}(\Theta - \theta_0)\right)\right) + \chi\left(\operatorname{sh}\frac{\chi}{2}(\Theta - \theta_0)\right)^{-1} \\
&\xi_{1,2} = a^0\exp\left(-\frac{\chi}{2}(\Theta - \theta_0)\right) + \frac{\eta_{1,2}^0}{\chi}\operatorname{sh}\frac{\chi}{2}(\Theta - \theta_0)
\end{aligned}
\tag{4.14}
$$

where $\eta_{1,2}^0$ are the values of $\eta_{1,2}$ at $\theta = \theta_0$; they are expressed in terms of $a^{0,f}$ and the parameters χ, k and $\Theta - \theta_0$ by formulae (4.14). Note that as $k \to \infty$ the coefficients satisfy the relation $\eta_1 \to \eta_2$, and moreover $\xi(\Theta) \to a^f$ for problem 1 (4.8).

Substituting $\eta_{1,2}$ for p into the formula for the control u^* we obtain constructions of the required open-loop and feedback controls, respectively

$$
\begin{aligned}
&u_p^* = -\eta(a^0, \theta - \theta_0, \Theta - \theta_0)\sin\psi \\
&u_s^* = -\eta(a, 0, \Theta - \theta)\sin\psi, \quad \theta = \varepsilon^2\psi
\end{aligned}
$$

The analogous elementary techniques outlined in Section 2 may be used to construct solutions with additional constraints on the control u, such as $u^- \leq u \leq u^+$. After the boundary-value problems have been reduced to the form of (2.5), the investigation is continued along the lines of the procedure in [3].

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